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LETTER TO THE EDITOR

The quantum rotor as an anomalous gauge theory

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Abstract. In the planar rotor, or the free particle on a circle, if we ignore the contributions to the propagator coming from paths belonging to non-trivial homotopy classes, we end up with a quantum theory which does not realize a classical symmetry. Applying a formalism, recently developed by Harada and Tsutsui and by Schaposnik, Babelon and Vialet in the context of anomalous gauge theories (AGT), will permit us to find the appropriated Wess–Zumino functional, restoring in this way the classical symmetry. The Wess–Zumino field is identified as a topological charge, the winding number. Based on this result a new construction for AGT is suggested which is anomaly free.

In recent years the study of anomalous gauge theories (AGT) has attracted a great deal of attention, mainly after the proposal by Polyakov [1] and, independently, by Faddeev and Shatashvili [2] that such theories could be consistently quantized with the inclusion of a compensator field which had been decoupled at classical level. Particularly for the case of gauge fields interacting with chiral fermions, it was believed that the only possible quantization scheme would be achieved by carefully adjusting the fermionic content of the theory [3]. On the other hand, Jackiw and Rajaraman (JR) [4], investigating the chiral version of the two-dimensional Schwinger model (CSM), found that in spite of the presence of an anomaly in the gauge current this theory was consistent and unitary.

Harada and Tsutsui (HT) [5] and Schaposnik, Babelon and Vialet (SBV) [6] proposed then, following the spirit of [1], that the unitarity in the JR model was a consequence of the presence of a Wess–Zumino (WZ) functional [7], which ultimately cancelled the anomaly. The CSM was then to be viewed as a gauge fixed version (called unitary gauge) of a truly gauge invariant theory [8]. To perform their analysis these authors made use of the Faddeev–Popov identity (FPI) [9] which would eventually produce the desired WZ functional. This technique was later extended outside the two-dimensional domain by Harada and Tsutsui [10]. To my understanding the FPI is just a trick which permits one to avoid the analysis of the existing constraints of the theory, as has been done by Faddeev [11] and Senjanovic [12]. Moreover the FPI is a device especially designed to fix the gauge in non-anomalous theories. HT and SBV, on the other hand, used it to 'unfix' the gauge of a theory that has had its gauge degree of freedom fixed by nature or by a wrong set of quantization rules. Dangerous, is the least one can say of its use in an AGT which does not possess that symmetry. An alternative approach [13] would be to look for a 'modified' theory, with the same

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underlying physics, but where the original second-class constraints had been tuned into first-class, signalling the recovery of gauge invariance. It is somewhat puzzling that the approach of [5] and [6] gives the right answer and to my knowledge no attempt has been made yet to justify this procedure and clarify the physical role played by the WZ term found by HT and SBV in the CSM. To shed some light on this matter one's attitude must be to look for some well understood model which would support their formalism. It is exactly this last point that we are proposing to do in this communication.

Before starting it is only fair to advise the reader about the speculative nature of the ideas presented in this letter, the reason being that the conclusions we have taken were based on results found on a quantum mechanical model with an artificially produced 'global anomaly' and not due to regularization of infinitely many degrees of freedom of a truly quantum field theory. Whether or not one can extend, legitimately, the conclusions of this 'fake anomaly' over the universe of anomalously broken gauge theories is unclear to us at the present, but we think that the parallelism between these two systems is very appealing and deserving of more attention.

Our approach here is the following: working with the 'amputated quantum rotor' [14], where the contributions of all the non-trivial topological sectors are ignored, will lead to a theory that mimics in some aspects an AGT. An obvious symmetry possessed by the classical system when it rounds S^1 an integer number of times is lost in the quantized model. Applying the HT-SBV approach will then cancel the 'anomaly' and produce the correct answer. It will be possible then to identify the WZ field with the topological charge of this theory, i.e. the numbers of turns around S^1. In view of these results one is then naturally led to consider the reverse argument and look at the anomaly of an AGT as a result of a non-trivial topology in its group manifold, which could be caused by a non-simply-connected structure. We will show that all the known results will follow from this argument and illustrate it with the CSM.

To compute the propagator for the quantum rotor is a simple quantum mechanical exercise. However, if we ignore all the non-trivial topological sectors, we end up with the following propagator:

$$K(\phi) \equiv e^{iW(\phi)} = \sqrt{\frac{1}{2\pi^2 i}}\, e^{i\phi^2/2\pi}. \tag{1}$$

Here ϕ is a compact angle variable (modulo 2π) traversed by a particle of unit mass per unit time. While the classical problem possesses a symmetry when ϕ rounds the ring an integer number of times, its (incorrect) quantum counterpart does not share the same symmetry, i.e.

$$K(\phi) \to K(\phi + 2n\pi) \neq K(\phi). \tag{2}$$

The system is then said to be anomalous. Let us introduce the following notation:

$$\begin{aligned} \phi^n &= \phi + 2\pi n \\ \phi^{n^{-1}} &= \phi - 2\pi n \\ K_n(\phi) &= K(\phi^n) = K(\phi + 2\pi n). \end{aligned} \tag{3}$$

The vacuum functional for this theory, defined as

$$Z = \int \mathscr{D}\phi\, e^{iW(\phi)} \tag{4}$$

is not invariant under the transformation $\phi \to \phi^n$, which we will take license to call 'gauge transformation'. Following HT and SBV we resolve the identity in the Faddeev-Popov way

$$1 = \int \mathscr{D}n \Delta[\phi] \delta F[\phi^n] \tag{5}$$

with obvious notation for the symbols introduced. Inserting this result into Z, changing the order of integration and relabelling the field ϕ we find

$$Z = \int \mathscr{D}\phi \Delta[\phi] \delta F[\phi] \, e^{i\tilde{W}(\phi)} \tag{6}$$

where

$$e^{i\tilde{W}(\phi)} = \int \mathscr{D}n \exp[i W(\phi^{n^{-1}})]. \tag{7}$$

Contrary to the case above, the functional $\tilde{W}(\phi)$ is invariant under the transformations $\phi \to \phi^n$

$$\begin{aligned} e^{i\tilde{W}(\phi^k)} &= \int \mathscr{D}n \exp[i W(\phi^{-(n-k)})] \\ &= \int \mathscr{D}(n-k) \exp[i W(\phi^{-(n-k)})] \\ &= e^{i\tilde{W}(\phi)}. \end{aligned} \tag{8}$$

In the HT-SBV language the system has recovered its 'gauge invariance' due to the presence of a WZ functional that has been extracted from the functional measure. HT called $W(\phi^{n^{-1}})$ a *standard action*, defining the WZ functional by

$$W(\phi^{n^{-1}}) = W(\phi) + \alpha(\phi, n^{-1}). \tag{9}$$

In our simple example the WZ term is found to be

$$\alpha(\phi, n^{-1}) = 2\pi n^2 - 2n\phi \tag{10}$$

and the 'gauge invariant' functional $\tilde{W}(\phi)$ becomes

$$\begin{aligned} e^{i\tilde{W}(\phi)} &= e^{iW(\phi)} \int \mathscr{D}n \, e^{i\alpha(\phi, n^{-1})} \\ &= e^{iW(\phi)} \theta_3(\phi, 2) \end{aligned} \tag{11}$$

where $\theta_3(z, t)$ is the third Jacobi theta-function [15] defined as

$$\theta(z, t) = \sum_{n=-\infty}^{\infty} e^{i(\pi t n^2 + 2nz)}. \tag{12}$$

The invariance of $\tilde{W}(\phi)$ under the gauge transformations is then guaranteed by the well known property

$$\theta_3(z + t\pi, t) = e^{-i\pi t - 2iz} \theta_3(z, t). \tag{13}$$

The physical situation in which a magnetic flux is confined in a region encircled by S^1 can be easily incorporated in our problem if we generalize the gauge invariant propagator in the following way

$$\tilde{K}[\phi] \to \tilde{K}_f[\phi] = \int \mathscr{D}n f(n) K_{-n}[\phi] \tag{14}$$

where $f(n)$ is a yet unknown field. Next we weaken the gauge invariance of $\tilde{K}_f[\phi]$ to be†

$$\tilde{K}_f[\phi] \to \tilde{K}_f[\phi^m] = g(m)\tilde{K}_f[\phi]. \tag{15}$$

Combining these two conditions we conclude that

$$f(n) = g(n) = \mathrm{e}^{\mathrm{i}n\delta} \tag{16}$$

which is just the amount of phase the particle picks up in each turn. Here δ is proportional to the flux intensity. The phenomenon depicted above is analogous to the vacuum structure of the Yang–Mills theory where n is the instanton number and δ is the labelling of the vacua states.

One can recognize that the gauge invariant propagator $\tilde{K}[\phi]$ obtained when the contributions from the WZ functional were taken into account is the correct propagator for the quantum rotor when one 'remembers' to include the paths belonging to different homotopy classes. The field variable n playing the role of the WZ field in the HT-SBV approach is here identified with the winding number. It becomes clear then that the anomalous behaviour of $K(\phi)$ was exclusively due to our poor understanding of the system's topological structure.

In view of the above results one is led to consider that the same sort of mechanism could be responsible for the anomaly in AGT, i.e. the presence of an anomaly is just the result of our ignorance about the vacuum structure of the theory. Even if the theory is a non-anomalous one, what should we preferably do in order to quantize it? Should we select a specific gauge configuration (by gauge fixing), or consider an average over all field configurations? By adopting the second point of view we are left with a non-anomalous theory even if we start with an effective potential which is not gauge invariant (obtained by integrating out the interacting fermions, for example). Consider then the following vacuum functional:

$$Z = \int \mathscr{D}A \sum_g \mathrm{e}^{\mathrm{i}W_g(A)} \tag{17}$$

where $W_g(A^h) = W_{gh}(A) \neq W_g(A)$. The sum (functional) may be viewed either as a sum over distinct topological sectors of the vacuum or as an average over all field configurations as discussed above. The gauge invariance is automatically guaranteed by the invariance of the functional measure under the group of gauge transformations. Introducing the WZ functional as before we have

$$Z = \int \mathscr{D}A\, \mathrm{e}^{\mathrm{i}W[A] + \mathrm{i}W_{\mathrm{WZ}}[A]} \tag{18}$$

where

$$\mathrm{e}^{\mathrm{i}W_{\mathrm{WZ}}[A]} = \sum_g \mathrm{e}^{\mathrm{i}\alpha(A,g)}. \tag{19}$$

† This is just the usual statement that the quantum mechanical wavefunction can change by a phase when the system undergoes some transformation without affecting physical results.

The total functional $\tilde{W}[A] = W[A] + W_{WZ}[A]$ is certainly gauge invariant since each term in the right-hand side compensates for the gauge non-invariance of the other. This property can be nicely illustrated in the CSM, whose effective action is

$$W[A] = -\tfrac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{8\pi} A_\mu \left(a g^{\mu\nu} - (g+\varepsilon)^{\mu\lambda} \frac{\partial_\lambda \partial_\sigma}{[\,]} (g+\varepsilon)^{\nu\sigma} \right) A_\nu . \tag{20}$$

The WZ term is [13]

$$\alpha(A, \theta) = \frac{1}{4\pi} \left(\frac{a-1}{2} (\partial_\mu \theta)^2 - e\theta[(a-1)\partial_\mu A^\mu - \varepsilon^{\mu\nu} \partial_\nu A_\mu \right). \tag{21}$$

By integrating out the WZ field θ we obtain

$$W_{WZ}[A] = \frac{e^2}{8\pi} A_\mu \left(\frac{g^{\mu\nu}}{a-1} + \frac{\partial^\mu \partial^\nu}{[\,]} \frac{a^2+2a-2}{a-1} + \frac{\tilde{\partial}^\mu \partial^\nu + \partial^\mu \tilde{\partial}^\nu}{[\,]} \right) A_\nu . \tag{22}$$

Adding together $W[A]$ and $W_{WZ}[A]$ results in the functional

$$\tilde{W}[A] = -\tfrac{1}{4} F_{\mu\nu} \frac{1}{[\,]} \left([\,]^2 + \frac{e}{4\pi} \frac{a^2}{a-1} \right) F^{\mu\nu} \tag{23}$$

which is, as promised, gauge invariant.

A recent tentative attempt to extend Faddeev and Shatashvili's ideas into an anomalous model in four dimensions was made by Levy [16]. He criticizes the approach advocated by the authors of [2] in the sense that it (apparently) seems to be a mixed quantum classical procedure, since the compensator field to be included into the classical action is already of order $\hbar$. While that seems to be true in the canonical point of view, we think that the example presented here shows that the appearance of a WZ term is a consequence of the quantization process, i.e. the result of many inequivalent quantum realizations of the same classical theory.

We conclude by suggesting that the possible mechanism behind the structure proposed in this letter could be the ability that chiral fermions have of 'opening holes' in the gauge fields' configuration space, similar to a Dirac string. In the case of Dirac fermions, made of two (opposite) chiral fermions, the configuration space is again simply connected since each fermion cancels the effect of the other, unless they are coupled to the gauge field with different strengths [17]. Even though the gauge symmetry is restored, for the case of Dirac fermions, the sewing of the two chiral fermions together still leaves an observable effect in a form of an axial anomaly, since they must share the same Dirac sea.

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